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Modeling mortality at old age with time-varying parameters

Pavel Zimmermann

Department of Statistics and Probability, University of Economics, Prague, Czech Republic

ABSTRACT
Several models of old age mortality with time-varying parameters are expressed in a single formula. In these models, the existence of an age threshold above which mortality increases over time and below which mortality decreases over time is problematic. The conditions of appearance of this threshold are expressed and shown on logistic and exponential models with empirical data. The conditions of appearance of the threshold reflect actual situations in developed countries. Richards' curve avoids the appearance of the threshold with empirical data.

KEYWORDS
logistic model; mortality models; old age; Richards' curve

1. Introduction

Table 1 shows how empirical mortality data become scarce at old ages, which explains why mortality rates are usually extrapolated from those at younger ages.

Mortality rates decrease over time in developed countries. Changes over time are modeled by introducing time or by assuming that some parameters vary in time. Observations are still too scarce after 105 years of age to favor one model over another. I shall show that most models with two time-varying parameters can be expressed in a unique formula. For such models, an age threshold exists above which mortality increases over time and below which mortality decreases over time. I shall modify these models to avoid the existence of this threshold. I propose to use Richards' (1959) curve, which extends the logistic curve, to avoid the appearance of such a threshold.

2. Model

Time t and age x are continuous. The force of mortality at age x and time t is $\mu(t, x)$. The probability of dying between age $x - 0.5$ at time $t - 0.5$ and $x + 0.5$ at time $t + 0.5$ is denoted $q(t, x)$:

$$q(t, x) = 1 - \int_{-0.5}^{0.5} \exp(-\mu(t + s, x + s)) ds. \quad (1)$$

CONTACT Pavel Zimmermann zimmerp@vse.cz University of Economics, Prague, Winston Churchill Square 4, 130 67, Prague 3, Czech Republic.

Table 1. Population exposed to risk of death and total number of deaths for men above 100 years of age in years 2005 to 2009. Source: Adapted from The Human Mortality Database (n.d.).

	France		Denmark		Czech Republic	
Age	Deaths	Population	Deaths	Population	Deaths	Population
100	1792	3919	99	235	90	150
101	1144	2206	66	141	35	82
102	650	1185	40	80	28	46
103	350	630	25	40	16	24
104	181	328	10	17	9	11
105	93	174	4	10	4	4
106	54	85	4	4	2	1
107	27	39	3	2	0	0
108	13	17	1	1	0	0
109	6	6	0	1	0	0
110+	2	1	1	1	0	0

$D(t,x)$ is the total number of deaths at age older than $x-0.5$ and younger than $x+0.5$, occurring in the time interval $(t-0.5, t+0.5)$. $p(t,x)$ is the population of men exposed to risk of death at age older than $x-0.5$ and younger than $x+0.5$ in the time interval $(t-0.5, t+0.5)$, in person-years. $m(t,x)$ denotes the mortality rate defined as

$$m(t,x) = \frac{D(t,x)}{p(t,x)}. \tag{2}$$

Wilmoth et al. (2007) estimate q and m in discrete time and age in the protocol of the Human Mortality Database.

Most one-dimensional models of mortality at old ages with time varying parameters are of the form:

$$z(t,x) = g(\eta(t,x)), \tag{3}$$

where $z(t,x)$ is $\mu(t,x)$, $q(t,x)$, or $m(t,x)$, and g is a differentiable monotonically increasing function,

$$\eta(t,x) = a(t) + b(t)h(x), \tag{4}$$

where $a(t)$ and $b(t)$ are time-varying parameters and h is either the identity or the logarithm. Models covered by Eq. (3) are listed in Table 2. With the addition of a quadratic term, Eq. (3) also includes the model introduced by Coale and Kisker (1990):

Table 2. Models (originally applied for fixed time) covered by Eq. (3).

Specification	z	g	Reference
$\operatorname{logit}(\mu(t,x)) = a(t) + b(t)x$	$\mu(t,x)$	logistic	Thatcher et al. (1998)
$\ln(\mu(t,x)) = a(t) + b(t)x$	$\mu(t,x)$	exponential	Gompertz (1825)
$\operatorname{logit}(m(t,x)) = a(t) + b(t)x$	$m(t,x)$	logistic	Buettner (2002)
$\operatorname{logit}(q(t,x)) = a(t) + b(t)x$	$q(t,x)$	logistic	Heligman and Pollard (1980)
$\operatorname{logit}(\mu(t,x) - c) = a(t) + b(t)x$	$\mu(t,x) - c$	logistic	Thatcher (1999)
$\ln(m(t,x)) = a(t) + b(t)\ln(x)$	$m(t,x)$	exponential	Boleslawski and Tabeau (2001)

$$m(t,x) = \exp\left(a(t) + b(t)x + c(t)x^2\right). \quad (5)$$

If a and b are linear functions of time t, Eq. (3) becomes

$$g^{-1}(z(t,x)) = k_1 + k_2 h(x) + k_3 t + k_4 h(x)t, \quad (6)$$

where k_j, $j = 1, 2, 3, 4$ are parameters.

Theorem 1. *In mortality models of* Eq. (3) *and (4), if the parameter $b(t)$ increases strictly in time t, there exists a $x^*(t) > 0$ such that $z(t,x)$ increases strictly in time for any $x > x^*(t)$.*
Proof. The partial derivative of $z(t,x)$ with respect to time is

$$\frac{\partial z(t,x)}{\partial t} = \frac{\partial g(\eta(t,x))}{\partial \eta(t,x)} \frac{\partial \eta(t,x)}{\partial t} = \frac{\partial g(\eta(t,x))}{\partial \eta(t,x)} (a'(t) + b'(t)h(x)), \quad (7)$$

where $a'(t)$ and $b'(t)$ are the derivatives of a and b with respect to t. As g monotonically increases with η, $z(x,t)$ increases in time if $a'(t) + b'(t)h(x) > 0$. If $b(t)$ increases strictly in time, there exists some $x^*(t) > 0$ for each fixed t, such that

$$h(x)b'(t) > -a'(t) \quad (8)$$

for $x > x^*(t)$, as $h(x)b'(t)$ increases strictly with age and $-a'(t)$ is constant. □

For almost all ages with a sufficient total number of observations, mortality rates have been decreasing over time in most European countries. Theorem 1 states that Eq. (3) describes increasing mortality over time at old ages, which is contradicted by the data. Such a threshold can occur at the age of 95 years and can cause counterintuitive predictions.

3. Logistic and exponential models

I use data for men from 1980 to 2011 in eight European countries, taken from the Human Mortality Database (http://www.mortality.org). My programs in the statistical software R are available at http://pages.vse.cz/zimmerp/publ/pred_ha_scrpt.zip. Data are collected over discrete one-year time intervals with midpoints t_i and age intervals with midpoints x_j. Similarly as in Brouhns et al. (2002), I assume that the total number of deaths follows a Poisson distribution:

$$D(t_i, x_j) \sim \text{Poisson}\left(p(t_i, x_j)\mu(t_i, x_j)\right). \quad (9)$$

I compute the maximum likelihood to fit population size and total number of deaths observed in each time t_i. The likelihood in Eq. (9) is:

$$L(a(t_i), b(t_i)) = \prod_{j \in V} \exp(-p(t_i, x_j)\mu(t_i, x_j)) \frac{p(t_i, x_j)\mu(t_i, x_j)^{D(t_i, x_j)}}{D(t_i, x_j)!}, \quad (10)$$

where $V = \{j : x_{\min} \le x_j \le x_{\max}\}$. $x_{\max} = 100.5$ years is the oldest age I considered. I tested three different values for the youngest age $x_{\min}$: 65.5, 80.5, and 85.5 years. Maximizing L amounts to maximize its logarithm with constant terms omitted:

$$l(a(t_i), b(t_i)) = \sum_{j \in V} D(t_i, x_j) \ln(\mu(t_i, x_j)) - p(t_i, x_j)\mu(t_i, x_j). \quad (11)$$

Then I fit for the trends of estimated $a(t_i)$ and $b(t_i)$. In case of a and b linear with time t, derivatives in Eq. (8) are constant and $x^*(t)$ is independent of time. Table 3 presents the linear time trend of the parameter estimates for Denmark for both logistic and exponential functions, with $x_{\min} = 80.5$ years. All estimates are significant at 5%. The estimates of the growth parameters for the logistic function are $a'(t) = -0.0834$ and $b'(t) = 0.0009$. For age older than $x^* = -a'(t)/b'(t) = 96.5$ years, the estimated force of mortality increases in time (Figure 1 left). For the exponential curve, this age is also $x^* = 96.5$ years (Figure 1 right). Table 4 presents thresholds x^* for other countries and minimum ages $x_{\min}$.

The thresholds vary across countries as well as with minimum ages $x_{\min}$. For $x_{\min} = 65.5$ years, the threshold for all countries except Denmark is higher than for $x_{\min} = 80.5$ or 85.5 years. Mortality projections for pension funds or insurance companies often assume a maximum age of 110 to 120 years. For the Netherlands, Norway, Denmark, and the United Kingdom, for all three $x_{\min}$, and for Spain, Italy, and the Czech Republic for $x_{\min} = 80.5$ and 85.5, I estimated thresholds x^* from 94 to 103 years of age. The projections then result in a force of mortality increasing for a substantial part of the projected age range lying above the threshold, while for ages below the threshold, mortality decreases. This property of the projections is an anomaly, which should be rectified.

Table 3. Estimated linear time trends of parameters $a(t)$ and $b(t)$ for Denmark for the period 1980 to 2011 and $x_{\min} = 80.5$ years.

Curve	Parameter	Trend parameter	Estimate	Standard error
Logistic	$a(t)$	intercept	155.14	12.81
		slope	−0.08	0.01
	$b(t)$	intercept	−1.61	0.15
		slope	0.00	0.00
Exponential	$a(t)$	intercept	134.07	10.28
		slope	−0.07	0.01
	$b(t)$	intercept	−1.40	0.11
		slope	0.00	0.00

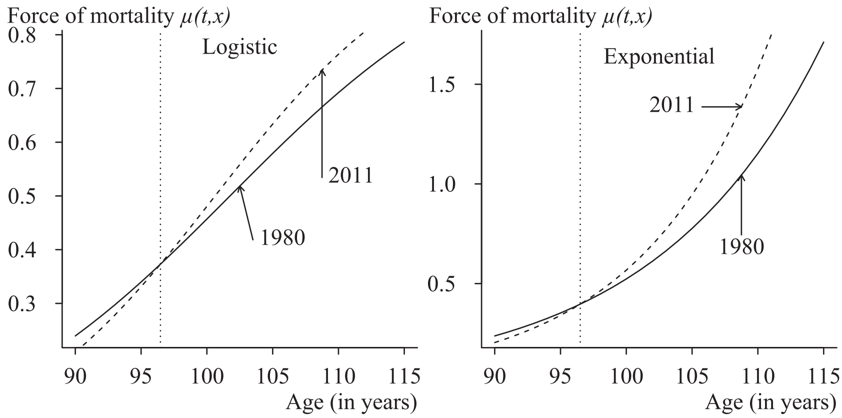

Figure 1. Estimated force of mortality in Denmark for calendar years 1980 and 2011 with logistic and exponential functions. The parameters $a(t)$ and $b(t)$ are smoothed linearly.

Table 4. Age threshold x^* over which the estimated force of mortality increases over time for selected countries and minimum ages considered for estimates.

		x^* in years		
Country	Curve	$x_{min} = 65.5$	$x_{min} = 80.5$	$x_{min} = 85.5$
the Netherlands	logistic	95.6	94.2	94.2
	exponential	94.7	94.1	94.5
Norway	logistic	96.6	94.8	95.4
	exponential	96.1	94.8	95.4
Denmark	logistic	98.4	96.5	100.1
	exponential	97.6	96.5	99.4
United Kingdom	logistic	112.1	103.3	102.5
	exponential	107.8	100.9	100.6
Spain	logistic	117.2	103.5	100.0
	exponential	112.4	101.2	99.0
Italy	logistic	110.5	105.6	106.3
	exponential	107.5	102.7	102.9
Czech Republic	logistic	184.1	108.7	108.4
	exponential	134.5	103.4	103.2
France	logistic	132.5	109.2	108.3
	exponential	121.0	105.1	104.2

4. Erasing the threshold

To avoid the existence of an age over which the model drives the force of mortality to increase over time, one possibility is to assume that mortality at age older than $x^*(t)$ is constant over time. Another possibility is to assume a maximum age $\omega(t)$ for which $q(\omega(t)) = 1$. For example Denuit and Goderniaux (2005) assume a quadratic trend with $\omega(t) = 130$ years for all dates t. Even a $\omega(t)$ varying over time does not guarantee that mortality at all ages decreases over time. As Table 4 shows, lowering x_{min} often increase $x^*(t)$. For France and the Czech Republic, thresholds for $x_{min} = 65.5$ are located from 120 to 185 years, so the threshold is avoided for the ages relevant for practical tasks.

Changing the specification of $\mu(t,x)$ can also be a solution. Richards (1959) proposed:

$$\mu(t,x) = (1 + a(t)\exp(-b(t)(x - c(t))))^{\frac{-1}{a(t)}}, \tag{12}$$

where $a(t)$, $b(t)$, and $c(t)$ are real time-dependent functions. This curve is not symmetric around its inflection point. The inequality

$$\frac{\partial\mu(t,x)}{\partial t} < 0 \tag{13}$$

has no closed form solution. A time-varying lower bound $\tilde{x}(t)$, an age above which the modeled force of mortality is bound to decrease over time, exists at each time t. The conditions observed in Europe lead us to assume that:

$$a(t) > 0 \text{ and } b(t) > 0 \text{ for all } t > 0, \tag{14}$$

$$a'(t) < 0 \text{ and } b'(t) < 0 \text{ for all } t > 0. \tag{15}$$

Theorem 2. *Under conditions (14) and (15), Richards' curve with time-varying parameters of Eq. (12) decreases over time for all ages $x > \tilde{x}(t)$, where*

$$\tilde{x}(t) = b(t)\frac{c'(t)}{b'(t)} + c(t). \tag{16}$$

The proof is in the Appendix. If $b(t)$ and $c(t)$ are linear in time t, $\tilde{x}(t)$ is also linear in time. For age under the lower bound $\tilde{x}(t)$, the predicted force of mortality may not decrease over time. I now check numerically the absence of solution of the inequality $\frac{\partial\mu(t,x)}{\partial t} > 0$ on the region bounded by $x_{\min} \leq x \leq \tilde{x}(t)$ and on the considered time interval.

I smoothed the observed mortality rates $m(t,x)$ for each country listed in Table 4 and fitted a Richards' function of Eq. (12) using ordinary least squares, assuming that in the centers of the age and time intervals, $\mu(t,x) \sim m(t,x)$. I used the R two-step procedure `locfit` documented in Loader and Liaw (2013). Local regression models are fitted at each point, which is then replaced by its forecast. In order to capture dispersion with ages, I used prior weights, each equal to the inverse of the variance of the estimator of the binomial ratio:

$$w(t,x) = \frac{p(t,x)}{m(t,x)(1 - m(t,x))}. \tag{17}$$

Figure 2 shows the estimated Richards' curve for men in France and Denmark in the years 1980 and 2011, with $x_{\min} = 80.5$ years. For France, the lower bound $\tilde{x}(t)$ is negative for any $t \leq 4995$ years. Theorem 2 stipulates that the undesirable increase of mortality over time does not occur for any positive age and

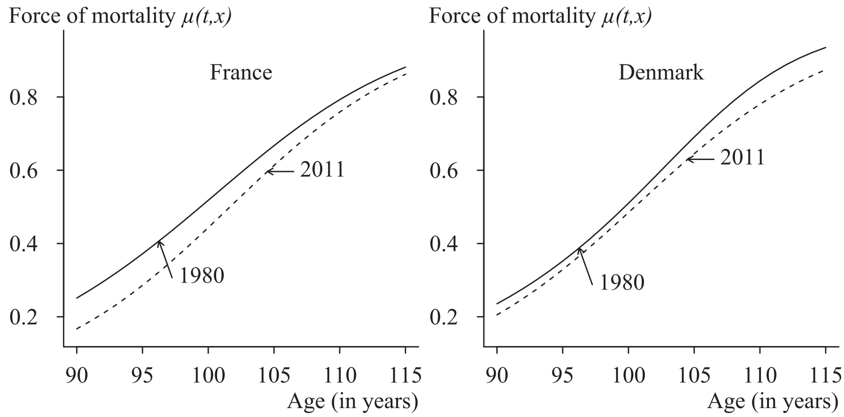

Figure 2. Estimated force of mortality at old age for the years 1980 and 2011 for French and Danish men, using Richards' function of Eq. (12). Parameters $a(t)$, $b(t)$, and $c(t)$ are smoothed by linear interpolation. $x_{\min} = 80.5$ years.

reasonable forecasting horizon, as $\mu(x, t)$ is decreasing for any age $x > \tilde{x}(t)$. For Denmark, the lower bound starts from $\tilde{x}(1980) = 107.9$ years and decreases to $\tilde{x}(2011) = 104.1$ years. $\mu(t, x)$ decreases over time for any $t \in [1980, 2011]$ and any $x \in [x_{\min}, \tilde{x}(t)]$, except very briefly around 1981 and around 100 years of age. According to Theorem 2, the estimated mortality should not increase for any age $x > \tilde{x}(t)$ either. For the Czech Republic and for Italy, the force of mortality never increases. For the Netherlands, Norway, Spain, and the United Kingdom, however, Richards' function fails to help the force of mortality from increasing. Figure 3 shows the estimated Richards' curves for Norwegian and Spanish men. For the Netherlands and Norway, the increase of mortality already appears at around 96 years of age. For Spain, mortality increases around 100 years of age. For the United Kingdom, the increase occurs around 110 years of age. The results are the same with $x_{\min} = 65.5$ years and $x_{\min} = 85.5$ years.

5. Conclusion

The existence in several common mortality models of a threshold above which forecast mortality rates increase over time is an anomaly. Applications indicate the threshold often ranges between 95 and 110 years. In pension insurance or other longevity risk products, this problem leads to inconsistent results. For some countries, using Richards' function avoids the undue increase of mortality over time. For some other countries, however, increasing mortality occurs even with Richards' function. So Richards' function solves the anomaly only partially.

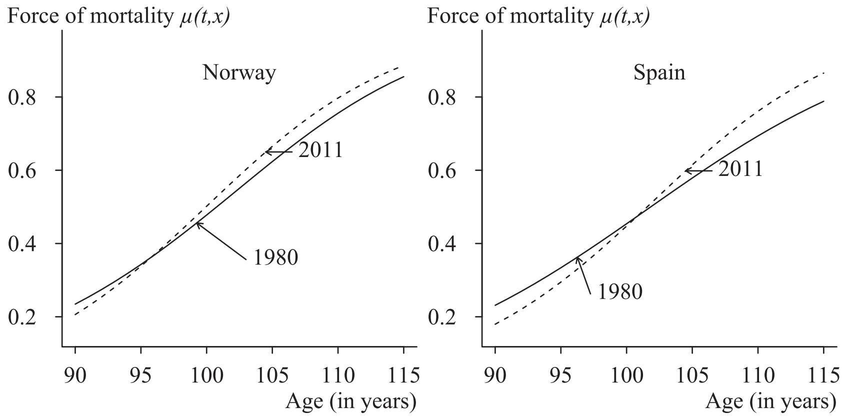

Figure 3. Estimated force of mortality at old age for the years 1980 and 2011 for Norwegian and Spanish men, using Richards' function of Eq. (12). Parameters $a(t)$, $b(t)$, and $c(t)$ are smoothed by linear interpolation. $x_{\min} = 80.5$ years.

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Appendix

Proof of Theorem 2

The derivative of Eq. (12) with respect to time is

$$\frac{\partial \mu(t,x)}{\partial t} = \frac{f_1(t,x) - f_2(t,x)}{a(t)^2 (a(t)\exp(-b(t)(x-c(t))) + 1)^{(1+1/a(t))}}, \tag{18}$$

where

$$f_1(t,x) = -a(t)^2(-b'(t)(x-c(t)) + b(t)c'(t))\exp(-b(t)(x-c(t))) \tag{19}$$

and

$$\begin{aligned} f_2(t,x) = {} & a'(t)(a(t)\exp(-b(t)(x-c(t)))- \\ & -\ln(\exp(-b(t)(x-c(t)))a(t)+1)(\exp(-b(t)(x-c(t)))a(t)+1)). \end{aligned} \tag{20}$$

As, under conditions of Eq. (14), $a(t) > 0$ for all t, the denominator of Eq. (18) is positive and I limit the analysis to the conditions under which the nominator in Eq. (18) is negative. The inequality $f_1(t,x) - f_2(t,x) < 0$ however have no analytic solution for x.

The function f_1 is not monotone with respect to age x. Under conditions of Eq. (14) and (15):

$$\lim_{x\to\infty} f_1(t,x) < 0. \tag{21}$$

For any t, $f_1(t,x)$ has one root at

$$\tilde{x}(t) = \frac{b(t)c'(t)}{b'(t)} + c(t). \tag{22}$$

Hence,

$$f_1(t,x) < 0 \text{ for } x > \tilde{x}(t). \tag{23}$$

Moreover,

$$\lim_{x\to\infty} f_2(t,x) = 0. \tag{24}$$

The derivative of $f_2(t,x)$ is

$$\frac{\partial f_2(t,x)}{\partial x} = a(t)a'(t)b(t)\ (\ln(\exp(-b(t)(x-c(t)) + 1))\exp(-b(t)(x-c(t)))). \tag{25}$$

Under the conditions of Eq. (14) and (15), the derivative is negative for all x and t. This property together with the limit in Eq. (24) implies that

$$f_2(t,x) > 0 \tag{26}$$

for any x and t. This together with Eq. (23) means that $f(t,x) < 0$ for all $x > \tilde{x}(t)$ and t.